Perturbative Aspects of QCD with Abelian Projection

1 Lagrangian for SU(2) QCD

For the purpose of perturbative study, a constant n^a is enough. We will take $\vec{n} = (0, 0, 1)$ in the SU(2) case.

1.1 Field Strength

Covariant derivative:

$$D_\mu \psi = (\partial_\mu - igG^\alpha_\mu T^\alpha)\psi, \quad [T^\alpha, T^\beta] = if^{\alpha\beta\gamma}T^\gamma$$

For the SU(2) Lie algebra $T^\alpha = \frac{\sigma^\alpha}{2}$. The gauge field combined with the matrix T^α is

$$\mathbf{G}_\mu = G^\alpha_\mu \frac{\sigma^\alpha}{2} = \frac{1}{2}\begin{pmatrix} G^3_\mu & G^1_\mu - iG^2_\mu \\ G^1_\mu + iG^2_\mu & -G^3_\mu \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A_\mu & \sqrt{2}X_\mu \\ \sqrt{2}X^*_\mu & -A_\mu \end{pmatrix}$$

where we have introduced

$$A_\mu = G^3_\mu, \quad \sqrt{2}X_\mu = G^1_\mu - iG^2_\mu.$$

Field strength tensor is

$$\mathbf{F}_{\mu\nu} = \frac{i}{g}[D_\mu . D_\nu] = \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu - ig[\mathbf{G}_\mu, \mathbf{G}_\nu]$$

It becomes, using new variables,

$$(\mathbf{F}_{\mu\nu}) = \frac{1}{2}\begin{pmatrix} F_{\mu\nu} - igX_{[\mu}X^*_{\nu]} & \sqrt{2}D_{[\mu}X_{\nu]} \\ \sqrt{2}D_{[\mu}X^*_{\nu]} & -F_{\mu\nu} + igX_{[\mu}X^*_{\nu]} \end{pmatrix}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu X_\nu = (\partial_\mu - igA_\mu)X_\nu, \quad D_\mu X^*_\nu = (\partial_\mu + igA_\mu)X^*_\nu$$

1.2 Lagrangian

The Lagrangian is $\mathcal{L}_0 = -\frac{1}{2}\mathrm{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu})$, i.e.,

$$\mathcal{L}_0 = -\frac{1}{2}\Big((\mathbf{F}_{\mu\nu})_{11}(\mathbf{F}^{\mu\nu})_{11} + 2(\mathbf{F}_{\mu\nu})_{12}(\mathbf{F}^{\mu\nu})_{21} + (\mathbf{F}_{\mu\nu})_{22}(\mathbf{F}^{\mu\nu})_{22}\Big)$$

Then we have

$$\mathcal{L}_0 = -\frac{1}{4}\Big(F_{\mu\nu} - igX_{[\mu}X^*_{\nu]}\Big)^2 - \frac{1}{2}(D^\mu X^\nu - D^\nu X^\mu)(D_\mu X^*_\nu - D_\nu X^*_\mu)$$

1.3 Gauge fixing: Covariant gauge

The Lagrangian has the form:

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D^{[\mu}X^{*\nu]}D_{[\mu}X_{\nu]} - \frac{1}{4}(-2ig)F^{\mu\nu}X_{[\mu}X^*_{\nu]} + \frac{1}{4}\left(X_{[\mu}X^*_{\nu]}\right)^2 \\ &= +\frac{1}{2}A^\mu\partial^2 A_\mu + \frac{1}{2}(\partial^\mu A_\mu)^2 + X^*_\nu(\eta^{\mu\nu}D^2 - [D^\mu, D^\nu])X_\mu + (D^\mu X_\mu)(D^\nu X^*_\nu) \\ &\quad + igF^{\mu\nu}X_\mu X^*_\nu + \frac{1}{4}\left(X_{[\mu}X^*_{\nu]}\right)^2\end{aligned}$$

We have ignored possible total derivative terms.

To quantize the system it is necessary to fix the gauge. Our choice of gauge fixings are the Feynman gauge for Abelian gauge fields A_μ and the background covariant) gauge for charged gauge field (chromon fields) X_μ. Thus, to the Lagrangian above, we add following gauge fixing terms

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - \frac{1}{\beta}|D^\mu X_\mu|^2.$$

We will not treat X as gluon fields but treat it as a charged vector field. Adding these two we have

$$\mathcal{L}_0 + \mathcal{L}_{\text{g.f.}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + (1 - \frac{1}{\beta})|D^\mu X_\mu|^2 + X^*_\mu(\eta^{\mu\nu}D^2 - 2igF^{\mu\nu})X_\nu + \frac{1}{4}\left(X_{[\mu}X^*_{\nu]}\right)^2$$

Gauge fixing should be accompanied with ghost terms. First let us split the gauge transformation of gauge fields in accordance of Abelian projection.

$$\delta \mathbf{G}_\mu = [D_\mu, \Lambda], \qquad \Lambda = \frac{1}{2}\begin{pmatrix} \lambda_3 & \sqrt{2}\lambda \\ \sqrt{2}\lambda^* & -\lambda_3 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} \partial_\mu \lambda_3 - ig(X_\mu \lambda^* - X_\mu^* \lambda) & \sqrt{2}(D_\mu \lambda + igX_\mu \lambda_3) \\ \sqrt{2}(D_\mu \lambda^* - igX_\mu^* \lambda_3) & -\partial_\mu \lambda_3 + ig(X_\mu \lambda^* - X_\mu^* \lambda) \end{pmatrix}$$

or

$$\delta A_\mu = \partial_\mu \lambda_3 - ig(X_\mu \lambda^* - X_\mu^* \lambda), \quad \delta X_\mu = D_\mu \lambda + igX_\mu \lambda_3$$

Now we consider variation of the gauges chosen:

$$\begin{aligned} \delta(\partial^\mu A_\mu) &= \partial^\mu(\partial_\mu \lambda_3 - ig(X_\mu \lambda^* - X_\mu^* \lambda)) \\ \delta(D^\mu X_\mu) &= D^\mu(D_\mu \lambda + igX_\mu \lambda_3) - ig[\partial_\mu \lambda_3 - ig(X_\mu \lambda^* - X_\mu^* \lambda)]X^\mu \\ &= D^\mu D_\mu \lambda + ig(D^\mu X_\mu)\lambda_3) - g^2(X_\mu \lambda^* - X_\mu^* \lambda)X^\mu \end{aligned}$$

Hence the ghost terms should be

$$\begin{aligned} \mathcal{L}_{\text{ghost}} = \bar{c}[&\partial^\mu(\partial_\mu c + igX_\mu^* c_+ - igX_\mu)c_-] \\ &+ \bar{c}_+[+ig(D^\mu X_\mu)c + (D^\mu D_\mu + g^2 X_\mu^* X^\mu)c_+ - g^2 X^\mu X_\mu c_- \\ &+ \bar{c}_-[-ig(D^\mu X_\mu)c - g^2 X^{*\mu} X_\mu^* c_+ + (D^\mu D_\mu + g^2 X_\mu^* X^\mu)c_-], \end{aligned}$$

denoting ghost fields, c, c_+, c_- and anti-ghost fields $\bar{c}, \bar{c}_+, \bar{c}_-$

1.4 Quark interactions

Lagrangian for quark sector is

$$\begin{aligned}\mathcal{L}_{\text{quark}} &= i\bar{\psi}\gamma^{\mu}D_{\mu}\psi - m\bar{\psi}\psi \\ &= \bar{\psi}\gamma^{\mu}[i\partial_{\mu} + g\frac{\sigma_3}{2} + \frac{g}{\sqrt{2}}\sigma_{12}X_{\mu} + \frac{g}{\sqrt{2}}\sigma_{21}X^{*}_{\mu}]\psi - m\bar{\psi}\psi\end{aligned}$$

where

$$\sigma_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

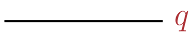
2 Feynman rules

2.1 Propagators

Here are propagators for photon (A) and charged vector particle (=X):

 A $$-i\frac{\eta_{\mu\nu}}{p^2 + i\epsilon} \quad \text{for } (A)$$

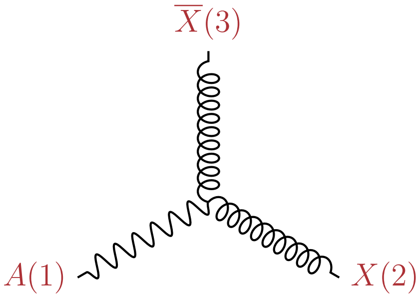
 X $$-i\frac{\eta_{\mu\nu}}{p^2 + i\epsilon} \quad \text{for } (X)$$



$$\frac{i}{\gamma^\mu p_\mu + i\epsilon} \quad \text{for (quark)}$$

2.2 3-point vertex functions

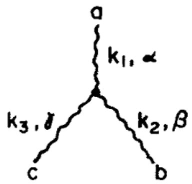
Here are three point vertices :

- 

$$= -2igp_1^{\mu_3}\eta_{\mu_1\mu_2} + 2igp_1^{\mu_2}\eta_{\mu_1\mu_3} + ig(p_3 - p_2)^{\mu_1}\eta_{\mu_2\mu_3}$$

- compare this with the original three point vertex

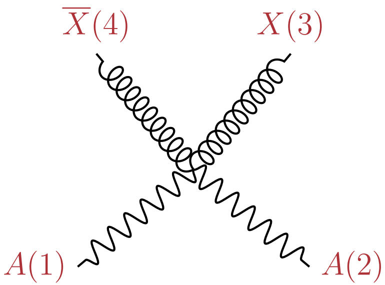
Three - Gluon Vertex



$$-ig_o f_{abc} \left[g^{\alpha\beta}(k_1-k_2)^{\gamma} + g^{\beta\gamma}(k_2-k_3)^{\alpha} + g^{\gamma\alpha}(k_3-k_1)^{\beta} \right]$$

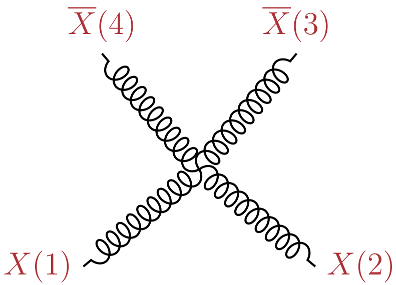
2.3 4-point vertex functions

Here are four point vertex with photons and charged vector particles

- 

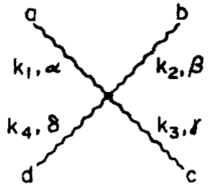
$$-2ig^2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$$

Here is a vertex involving four charged vector particles

- 

$$-ig^2[\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - 2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}]$$

- Four-Gluon Vertex



$$ig_0^2 \Big[f_{nac} f_{nbd} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}) + f_{nad} f_{nbc} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}) + f_{nab} f_{ncd} (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \Big]$$

a, α

- There are other diagrams describing quark-gluon and ghost-gluon interactions

3 Lagrangian for SU(3) QCD

3.1 Field Strength

Covariant derivative:

$$D_\mu \psi = (\partial_\mu - igG_\mu^\alpha T^\alpha)\psi, \quad [T^\alpha, T^\beta] = if^{\alpha\beta\gamma}T^\gamma$$

For the SU(3) Lie algebra $T^\alpha = \frac{\lambda^\alpha}{2}$.

There are two Abelian matrices T^3 and T^8

The gauge field combined with the matrix T^α is

$$\begin{aligned}
\mathbf{G}_\mu = G_\mu^\alpha \frac{\lambda^\alpha}{2} &= \frac{1}{2}\begin{pmatrix} G_\mu^3 + \frac{1}{\sqrt{3}}G_\mu^8 & G_\mu^1 - iG_\mu^2 & G_\mu^4 - iG_\mu^5 \\ G_\mu^1 + iG_\mu^2 & -G_\mu^3 + \frac{1}{\sqrt{3}}G_\mu^8 & G_\mu^6 - iG_\mu^7 \\ G_\mu^4 + iG_\mu^5 & G_\mu^6 + iG_\mu^7 & -\frac{2}{\sqrt{3}}G_\mu^8 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} A_\mu + \frac{1}{\sqrt{3}}B_\mu & \sqrt{2}X_\mu^+ & \sqrt{2}Y_\mu^+ \\ \sqrt{2}X_\mu^- & -A_\mu + \frac{1}{\sqrt{3}}B_\mu & \sqrt{2}Z_\mu^+ \\ \sqrt{2}Y_\mu^- & \sqrt{2}Z_\mu^- & -\frac{2}{\sqrt{3}}B_\mu \end{pmatrix}
\end{aligned}$$

where we have introduced

$$\begin{aligned}
A_\mu = G_\mu^3,\ B_\mu = G_\mu^8,\ \sqrt{2}X_\mu = G_\mu^1 - iG_\mu^2 \\
\sqrt{2}Y_\mu = G_\mu^4 - iG_\mu^5,\ \sqrt{2}Z_\mu = G_\mu^6 - iG_\mu^7.
\end{aligned}$$

Field strength tensor:

$$\mathbf{F}_{\mu\nu} = \frac{i}{g}[D_\mu . D_\nu] = \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu - ig[\mathbf{G}_\mu, \mathbf{G}_\nu]$$

Curl derivative part:

$$\mathbf{Der} = \frac{1}{2}\begin{pmatrix} A_{\mu\nu} + \frac{1}{\sqrt{3}}B_{\mu\nu} & \sqrt{2}\partial_{[\mu}X^+_{\nu]} & \sqrt{2}\partial_{[\mu}Y^+_{\nu]} \\ \sqrt{2}\partial_{[\mu}X^-_{\nu]} & -A_{\mu\nu} + \frac{1}{\sqrt{3}}B_{\mu\nu} & \sqrt{2}\partial_{[\mu}X^+_{\nu]} \\ \sqrt{2}\partial_{[\mu}Y^-_{\nu]} & \sqrt{2}\partial_{[\mu}Z^-_{\nu]} & -\frac{2}{\sqrt{3}}B_{\mu\nu} \end{pmatrix}$$

For calculation of the commutator part, we need

$$\left[\begin{pmatrix} A_\mu + \frac{1}{\sqrt{3}}B_\mu & \sqrt{2}X^+_\mu & \sqrt{2}Y^+_\mu \\ \sqrt{2}X^-_\mu & -A_\mu + \frac{1}{\sqrt{3}}B_\mu & \sqrt{2}Z^+_\mu \\ \sqrt{2}Y^-_\mu & \sqrt{2}Z^-_\mu & -\frac{2}{\sqrt{3}}B_\mu \end{pmatrix}, \begin{pmatrix} A_\nu + \frac{1}{\sqrt{3}}B_\nu & \sqrt{2}X^+_\nu & \sqrt{2}Y^+_\nu \\ \sqrt{2}X^-_\nu & -A_\nu + \frac{1}{\sqrt{3}}B_\nu & \sqrt{2}Z^+_\nu \\ \sqrt{2}Y^-_\nu & \sqrt{2}Z^-_\nu & -\frac{2}{\sqrt{3}}B_\nu \end{pmatrix}\right]$$

Commutator part: $\mathbf{C}$

$$
\begin{aligned}
\mathbf{C}_{11} &= \frac{1}{4}\Big(2X^+_{[\mu}X^-_{\nu]} + 2Y^+_{[\mu}Y^-_{\nu]}\Big)\\
\mathbf{C}_{12} &= \frac{1}{4}\Big(2\sqrt{2}A_{[\mu}X^+_{\nu]} + 2Y^+_{[\mu}Z^-_{\nu]}\Big)\\
\mathbf{C}_{13} &= \frac{1}{4}\Big(\sqrt{2}A_{[\mu}Y^+_{\nu]} + \sqrt{6}B_{[\mu}Y^+_{\nu]} + 2X^+_{[\mu}Z^+_{\nu]}\Big)\\
\mathbf{C}_{21} &= \frac{1}{4}\Big(-2\sqrt{2}A_{[\mu}X^-_{\nu]} - 2Y^-_{[\mu}Z^+_{\nu]}\Big)\\
\mathbf{C}_{22} &= \frac{1}{4}\Big(2X^-_{[\mu}X^+_{\nu]} + 2Z^+_{[\mu}Z^-_{\nu]}\Big)\\
\mathbf{C}_{23} &= \frac{1}{4}\Big(-\sqrt{2}A_{[\mu}Z^+_{\nu]} + \sqrt{6}B_{[\mu}Z^+_{\nu]} + 2X^-_{[\mu}Y^+_{\nu]}\Big)\\
\mathbf{C}_{31} &= \frac{1}{4}\Big(-\sqrt{2}A_{[\mu}Y^-_{\nu]} - \sqrt{6}B_{[\mu}Y^-_{\nu]} - 2X^-_{[\mu}Z^-_{\nu]}\Big)\\
\mathbf{C}_{32} &= \frac{1}{4}\Big(\sqrt{2}A_{[\mu}Z^-_{\nu]} - \sqrt{6}B_{[\mu}Z^-_{\nu]} - 2X^+_{[\mu}Y^-_{\nu]}\Big)\\
\mathbf{C}_{33} &= \frac{1}{4}\Big(2Y^-_{[\mu}Y^+_{\nu]} + 2Z^-_{[\mu}Z^+_{\nu]}\Big)
\end{aligned}
$$

Hence,

$$(\mathbf{F}_{\mu\nu})_{11} = \frac{1}{2}\Big(A_{\mu\nu} + \frac{1}{\sqrt{3}}B_{\mu\nu} - igX^+_{[\mu}X^-_{\nu]} - igY^+_{[\mu}Y^-_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{12} = \frac{1}{2}\Big(\sqrt{2}D_{[\mu}X^+_{\nu]} - igY^+_{[\mu}Z^-_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{13} = \frac{1}{2}\Big(\sqrt{2}D_{[\mu}Y^+_{\nu]} - igX^+_{[\mu}Z^+_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{21} = \frac{1}{2}\Big(\sqrt{2}D_{[\mu}X^-_{\nu]} - igZ^+_{[\mu}Y^-_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{22} = \frac{1}{2}\Big(-A_{\mu\nu} + \frac{1}{\sqrt{3}}B_{\mu\nu} + igX^+_{[\mu}X^-_{\nu]} - igZ^+_{[\mu}Z^-_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{23} = \frac{1}{2}\Big(\sqrt{2}D_{[\mu}Z^+_{\nu]} - igX^-_{[\mu}Y^+_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{31} = \frac{1}{2}\Big(\sqrt{2}D_{[\mu}Y^-_{\nu]} - igZ^-_{[\mu}X^-_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{32} = \frac{1}{2}\Big(\sqrt{2}D_{[\mu}Z^-_{\nu]} - igY^-_{[\mu}X^+_{\nu]}\Big)$$
$$(\mathbf{F}_{\mu\nu})_{33} = \frac{1}{2}\Big(-\frac{2}{\sqrt{3}}B_{\mu\nu} + igY^+_{[\mu}Y^-_{\nu]} + igZ^+_{[\mu}Z^-_{\nu]}\Big)$$

where the covariant derivatives are

$$D_\mu X^+_\nu = (\partial_\mu - igA_\mu)X^+_\nu; \quad D_\mu Y^+_\nu = (\partial_\mu - ig\frac{1}{2}A_\mu - ig\frac{\sqrt{3}}{2}B_\mu)Y^+_\nu;$$
$$D_\mu Z^+_\nu = (\partial_\mu + ig\frac{1}{2}A_\mu - ig\frac{\sqrt{3}}{2}B_\mu)Z^+_\nu$$

3.2 Lagrangian

The Lagrangian is $\mathcal{L}_0 = -\frac{1}{2}\mathrm{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu})$, i.e.,

$$\begin{aligned}\mathcal{L}_0 = -\frac{1}{2}\Big(& (\mathbf{F}_{\mu\nu})_{11}(\mathbf{F}^{\mu\nu})_{11} + (\mathbf{F}_{\mu\nu})_{12}(\mathbf{F}^{\mu\nu})_{21} + (\mathbf{F}_{\mu\nu})_{13}(\mathbf{F}^{\mu\nu})_{31} \\ & + (\mathbf{F}_{\mu\nu})_{21}(\mathbf{F}^{\mu\nu})_{12} + (\mathbf{F}_{\mu\nu})_{22}(\mathbf{F}^{\mu\nu})_{22} + (\mathbf{F}_{\mu\nu})_{23}(\mathbf{F}^{\mu\nu})_{32} \\ & + (\mathbf{F}_{\mu\nu})_{31}(\mathbf{F}^{\mu\nu})_{13} + (\mathbf{F}_{\mu\nu})_{32}(\mathbf{F}^{\mu\nu})_{23} + (\mathbf{F}_{\mu\nu})_{33}(\mathbf{F}^{\mu\nu})_{33}\Big)\end{aligned}$$

Then we have

$$
\begin{aligned}
\mathcal{L}_0 = -\frac{1}{2}\Bigg[& \frac{1}{4}\Big(A_{\mu\nu} + \frac{1}{\sqrt{3}}B_{\mu\nu} - igX^+_{[\mu}X^-_{\nu]} - igY^+_{[\mu}Y^-_{\nu]}\Big)^2 \\
& + \frac{1}{4}\Big(-A_{\mu\nu} + \frac{1}{\sqrt{3}}B_{\mu\nu} + igX^+_{[\mu}X^-_{\nu]} - igZ^+_{[\mu}Z^-_{\nu]}\Big)^2 \\
& + \frac{1}{4}\Big(-\frac{2}{\sqrt{3}}B_{\mu\nu} + igY^+_{[\mu}Y^-_{\nu]} + igZ^+_{[\mu}Z^-_{\nu]}\Big)^2 \\
& + \frac{1}{2}\Big(\sqrt{2}D^{[\mu}X^{+\nu]} - igY^{+[\mu}Z^{-\nu]}\Big)\Big(\sqrt{2}D_{[\mu}X^-_{\nu]} - igZ^+_{[\mu}Y^-_{\nu]}\Big) \\
& + \frac{1}{2}\Big(\sqrt{2}D^{[\mu}Y^{+\nu]} - igX^{+[\mu}Z^{+\nu]}\Big)\Big(\sqrt{2}D_{[\mu}Y^-_{\nu]} - igZ^-_{[\mu}X^-_{\nu]}\Big) \\
& + \frac{1}{2}\Big(\sqrt{2}D^{[\mu}Z^{+\nu]} - igX^{-[\mu}Y^{+\nu]}\Big)\Big(\sqrt{2}D_{[\mu}Z^-_{\nu]} - igY^-_{[\mu}X^+_{\nu]}\Big)\Bigg]
\end{aligned}
$$

Where the Abelian covariant derivatives are

$$
\begin{aligned}
& D_\mu X^+_\nu = (\partial_\mu - igA_\mu)X^+_\nu, \quad D_\mu Y^+_\nu = (\partial_\mu - ig\frac{1}{2}A_\mu - ig\frac{\sqrt{3}}{2}B_\mu)Y^+_\nu \\
& D_\mu Z^+_\nu = (\partial_\mu + ig\frac{1}{2}A_\mu - ig\frac{\sqrt{3}}{2}B_\mu)Z^+_\nu
\end{aligned}
$$

We divide the full Lagrangian into two parts, kinetic energy part and interaction part.

$$\mathcal{L}_0 = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}$$

3.3 Kinetic part

The kinetic part of the Lagrangian (including all of the magnetic terms):

$$\begin{aligned}
\mathcal{L}_{\text{kin}} = &-\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}D^{[\mu}X^{-\nu]}D_{[\mu}X^{+}_{\nu]} + (X \to Y) + (X \to Z) \\
&-\frac{1}{4}(-ig)A^{\mu\nu}\big(2X^{+}_{[\mu}X^{-}_{\nu]} + Y^{+}_{[\mu}Y^{-}_{\nu]} - Z^{+}_{[\mu}Z^{-}_{\nu]}\big) \\
&-\frac{1}{4}(-ig)\frac{1}{\sqrt{3}}B^{\mu\nu}\big(Y^{+}_{[\mu}Y^{-}_{\nu]} + Z^{+}_{[\mu}Z^{-}_{\nu]}\big) \\
= &+\frac{1}{2}A^{\mu}\partial^2 A_{\mu} + \frac{1}{2}(\partial^{\mu}A_{\mu})^2 + \frac{1}{2}B^{\mu}\partial^2 B_{\mu} + \frac{1}{2}(\partial^{\mu}B_{\mu})^2 \\
&+ X^{-}_{\nu}(\eta^{\mu\nu}D^2 - [D^{\mu}, D^{\nu}])X^{+}_{\mu} + (D^{\mu}X^{+}_{\mu})(D^{\nu}X^{-}_{\nu}) + (X \to Y) + (X \to Z) \\
&- (-ig)A^{\mu\nu}\big(X^{+}_{\mu}X^{-}_{\nu} + \frac{1}{2}Y^{+}_{\mu}Y^{-}_{\nu} - \frac{1}{2}Z^{+}_{\mu}Z^{-}_{\nu}\big) \\
&- (-ig)\frac{\sqrt{3}}{2}B^{\mu\nu}\big(Y^{+}_{\mu}Y^{-}_{\nu} + Z^{+}_{\mu}Z^{-}_{\nu}\big)
\end{aligned}$$

We have ignored possible total derivative terms.

$$
\begin{aligned}
\mathcal{L}_{\rm kin} = & \frac{1}{2}A^\mu \partial^2 A_\mu + \frac{1}{2}(\partial^\mu A_\mu)^2 + \frac{1}{2}B^\mu \partial^2 B_\mu + \frac{1}{2}(\partial^\mu B_\mu)^2 \\
& + X_\mu^- (\eta^{\mu\nu} D^2 - 2igA^{\mu\nu}) X_\nu^+ + (D^\mu X_\mu^+)(D^\nu X_\nu^-) \\
& + Y_\mu^- (\eta^{\mu\nu} D^2 - igA^{\mu\nu} - ig\sqrt{3}B^{\mu\nu}) Y_\nu^+ + (D^\mu Y_\mu^+)(D^\nu Y_\nu^-) \\
& + Z_\mu^- (\eta^{\mu\nu} D^2 + igA^{\mu\nu} - ig\sqrt{3}B^{\mu\nu}) Z_\nu^+ + (D^\mu Z_\mu^+)(D^\nu Z_\nu^-)
\end{aligned}
$$

To quantize the system it is necessary to fix the gauge. Our choice of gauge fixings are the Feynman gauge for two Abelian gauge fields A_μ and B_μ and background gauge for three charged gauge fields (chromon fields) X_μ, Y_μ and Z_μ. Thus, to the Lagrangian above, we add following terms

$$
\mathcal{L}_{\rm g.f.} = -\frac{1}{2}(\partial^\mu A_\mu)^2 - \frac{1}{2}(\partial^\mu B_\mu)^2 - |D^\mu X_\mu^+|^2 - |D^\mu Y_\mu^+|^2 - |D^\mu Z_\mu^+|^2
$$

We will not treat X, Y, Z as gluon fields but treat them as three charged vector fields. Final form of the Lagrangian for our $U(1) \times U(1)$ is

$$
\begin{aligned}
\mathcal{L}_{\rm kin} + \mathcal{L}_{\rm g.f.} = & -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}(\partial^\mu A_\mu)^2 - \frac{1}{2}(\partial^\mu B_\mu)^2 \\
& + \overline{X}_\mu (\eta^{\mu\nu} D^2 - 2igA^{\mu\nu}) X_\nu + \overline{Y}_\mu (\eta^{\mu\nu} D^2 - igA^{\mu\nu} - ig\sqrt{3}B^{\mu\nu}) Y_\nu \\
& + \overline{Z}_\mu (\eta^{\mu\nu} D^2 + igA^{\mu\nu} - ig\sqrt{3}B^{\mu\nu}) Z_\nu
\end{aligned}
$$

where we have changed our notation slightly,

$$
X = X^+, \quad \overline{X} = X^-, \ etc.
$$

3.4 Interaction part

$$
\begin{aligned}
\mathcal{L}_{\rm int} = \frac{g^2}{8}\Big\{&(\overline{X}_{[\mu}X_{\nu]} + \overline{Y}_{[\mu}Y_{\nu]})^2 + (\overline{X}_{[\mu}X_{\nu]} - \overline{Z}_{[\mu}Z_{\nu]})^2 + (\overline{Y}_{[\mu}Y_{\nu]} + \overline{Z}_{[\mu}Z_{\nu]})^2 \\
&+ 2\overline{Z}^{[\mu}Y^{\nu]}\overline{Y}_{[\mu}Z_{\nu]} + 2X^{[\mu}Z^{\nu]}\overline{Z}_{[\mu}\overline{X}_{\nu]} + 2\overline{X}^{[\mu}Y^{\nu]}\overline{Y}_{[\mu}X_{\nu]}\Big\} \\
- ig\frac{\sqrt{2}}{4}&\Big\{D_{[\mu}X_{\nu]}\overline{Y}^{[\mu}Z^{\nu]} + D_{[\mu}Y_{\nu]}\overline{X}^{[\mu}\overline{Z}^{\nu]} + D_{[\mu}Z_{\nu]}X^{[\mu}\overline{Y}^{\nu]} - (\text{c.c.})\Big\} \\
= -ig\frac{\sqrt{2}}{2}&\Big\{D_\mu X_\nu(\overline{Y}^\mu Z^\nu - \overline{Y}^\nu Z^\mu) + D_\mu Y_\nu(\overline{X}^\mu\overline{Z}^\nu - \overline{X}^\nu\overline{Z}^\mu) + D_\mu Z_\nu(X^\mu\overline{Y}^\nu - X^\nu\overline{Y}^\mu) - (\text{c.c.})\Big\} \\
+ \frac{g^2}{8}&\Big\{(\overline{X}_{[\mu}X_{\nu]} + \overline{Y}_{[\mu}Y_{\nu]})^2 + (\overline{X}_{[\mu}X_{\nu]} - \overline{Z}_{[\mu}Z_{\nu]})^2 + (\overline{Y}_{[\mu}Y_{\nu]} + \overline{Z}_{[\mu}Z_{\nu]})^2 \\
&+ 2\overline{Z}^{[\mu}Y^{\nu]}\overline{Y}_{[\mu}Z_{\nu]} + 2X^{[\mu}Z^{\nu]}\overline{Z}_{[\mu}\overline{X}_{\nu]} + 2\overline{X}^{[\mu}Y^{\nu]}\overline{Y}_{[\mu}X_{\nu]}\Big\}
\end{aligned}
$$

Expanding the 2nd part, we have

$$\begin{aligned}
\mathcal{L}_{\text{int, 2}} &= \frac{g^2}{4}\Big\{(\overline{X}_{[\mu}X_{\nu]})^2 + (\overline{Y}_{[\mu}Y_{\nu]})^2 + (\overline{Z}_{[\mu}Z_{\nu]})^2 + \overline{X}_{[\mu}X_{\nu]}\overline{Y}^{[\mu}Y^{\nu]} - \overline{X}_{[\mu}X_{\nu]}\overline{Z}^{[\mu}Z^{\nu]} + \overline{Z}_{[\mu}Z_{\nu]}\overline{Y}^{[\mu}Y^{\nu]} \\
&\quad + \overline{Z}^{[\mu}Y^{\nu]}\overline{Y}_{[\mu}Z_{\nu]} + X^{[\mu}Z^{\nu]}\overline{Z}_{[\mu}\overline{X}_{\nu]} + \overline{X}^{[\mu}Y^{\nu]}\overline{Y}_{[\mu}X_{\nu]}\Big\} \\
&= \frac{g^2}{2}\Big\{\overline{X}^2X^2 - (\overline{X}\cdot X)^2 + \overline{Y}^2Y^2 - (\overline{Y}\cdot Y)^2 + \overline{Z}^2Z^2 - (\overline{Z}\cdot Z)^2 \\
&\quad + 2\overline{X}\cdot\overline{Y}X\cdot Y + 2\overline{X}\cdot Z\overline{Z}\cdot X + 2\overline{Y}\cdot\overline{Z}Y\cdot Z \\
&\quad - \overline{X}\cdot Y\overline{Y}\cdot X - \overline{X}\cdot\overline{Z}X\cdot Z - \overline{Y}\cdot Z\overline{Z}\cdot Y \\
&\quad - \overline{X}\cdot X\overline{Y}\cdot Y - \overline{X}\cdot X\overline{Z}\cdot Z - \overline{Y}\cdot Y\overline{Z}\cdot Z\Big\}
\end{aligned}$$

Lagrangian for gluons (neurons and chromons) is

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}(\partial^\mu A_\mu)^2 - \frac{1}{2}(\partial^\mu B_\mu)^2 + \overline{X}_\mu(\eta^{\mu\nu}D^2 - 2igA^{\mu\nu})X_\nu \\
&+ \overline{Y}_\mu(\eta^{\mu\nu}D^2 - igA^{\mu\nu} - ig\sqrt{3}B^{\mu\nu})Y_\nu + \overline{Z}_\mu(\eta^{\mu\nu}D^2 + igA^{\mu\nu} - ig\sqrt{3}B^{\mu\nu})Z_\nu \\
&- ig\frac{\sqrt{2}}{2}\Big\{D_\mu X_\nu(\overline{Y}^\mu Z^\nu - \overline{Y}^\nu Z^\mu) + D_\mu Y_\nu(\overline{X}^\mu \overline{Z}^\nu - \overline{X}^\nu \overline{Z}^\mu) + D_\mu Z_\nu(X^\mu \overline{Y}^\nu - X^\nu \overline{Y}^\mu) - (\text{c.c.})\Big\} \\
&\quad + \frac{g^2}{2}\Big\{\overline{X}^2 X^2 - (\overline{X}\cdot X)^2 + \overline{Y}^2 Y^2 - (\overline{Y}\cdot Y)^2 + \overline{Z}^2 Z^2 - (\overline{Z}\cdot Z)^2 \\
&\quad + 2\overline{X}\cdot\overline{Y}X\cdot Y + 2\overline{X}\cdot Z\overline{Z}\cdot X + 2\overline{Y}\cdot\overline{Z}Y\cdot Z \\
&\quad - \overline{X}\cdot Y\overline{Y}\cdot X - \overline{X}\cdot\overline{Z}X\cdot Z - \overline{Y}\cdot Z\overline{Z}\cdot Y \\
&\quad - \overline{X}\cdot X\overline{Y}\cdot Y - \overline{X}\cdot X\overline{Z}\cdot Z - \overline{Y}\cdot Y\overline{Z}\cdot Z\Big\}
\end{aligned}
$$

Do not forget

$$
\begin{aligned}
&D_\mu X_\nu = (\partial_\mu - igA_\mu)X_\nu, \quad D_\mu Y_\nu = (\partial_\mu - ig\frac{1}{2}A_\mu - ig\frac{\sqrt{3}}{2}B_\mu)Y_\nu, \\
&D_\mu Z_\nu = (\partial_\mu + ig\frac{1}{2}A_\mu - ig\frac{\sqrt{3}}{2}B_\mu)Z_\nu
\end{aligned}
$$

4 Feynman rules

4.1 Propagators

Here are propagators for two photons (A, B) and three charged vector particles (=X, Y, Z):

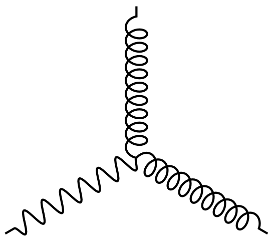
 A, B $\qquad -i\dfrac{\eta_{\mu\nu}}{p^2+i\epsilon}$ for (A,B)

 X, Y, Z $\qquad -i\dfrac{\eta_{\mu\nu}}{p^2+i\epsilon}$ for (X,Y,Z)

4.2 3-point vertex functions

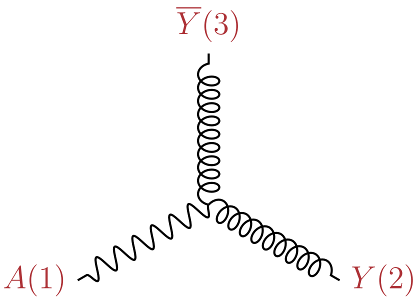
Here are 7 different three point vertices (7):

- $\overline{X}(3)$

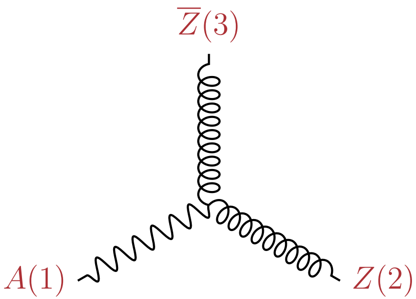
  

  $A(1)$ $\quad X(2)$

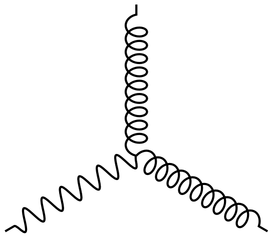
  $$= -2igp_1^{\mu_3}\eta_{\mu_1\mu_2} + 2igp_1^{\mu_2}\eta_{\mu_1\mu_3} + ig(p_3-p_2)^{\mu_1}\eta_{\mu_2\mu_3}$$

- 

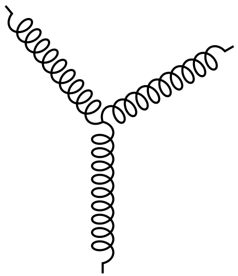
$$= \frac{1}{2}[-2igp_1^{\mu_3}\eta_{\mu_1\mu_2} + 2igp_1^{\mu_2}\eta_{\mu_1\mu_3} + ig(p_3 - p_2)^{\mu_1}\eta_{\mu_2\mu_3}]$$

- 

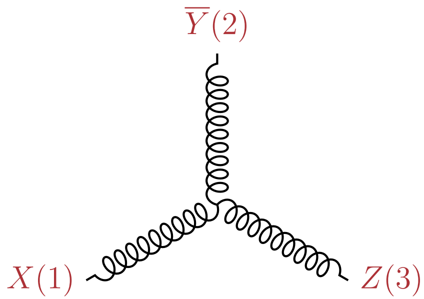
$$= -\frac{1}{2}[-2igp_1^{\mu_3}\eta_{\mu_1\mu_2} + 2igp_1^{\mu_2}\eta_{\mu_1\mu_3} + ig(p_3 - p_2)^{\mu_1}\eta_{\mu_2\mu_3}]$$

- 

$$= \frac{\sqrt{3}}{2}[-2igp_1^{\mu_3}\eta_{\mu_1\mu_2} + 2igp_1^{\mu_2}\eta_{\mu_1\mu_3} + ig(p_3 - p_2)^{\mu_1}\eta_{\mu_2\mu_3}]$$

- 

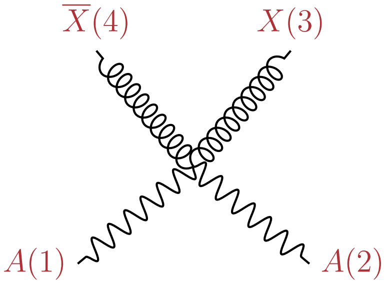
$$= \frac{1}{\sqrt{2}}[-ig(p_1 - p_2)^{\mu_3}\eta_{\mu_1\mu_2} - ig(p_2 - p_3)^{\mu_1}\eta_{\mu_2\mu_3} - ig(p_3 - p_1)^{\mu_2}\eta_{\mu_1\mu_3}]$$

- 

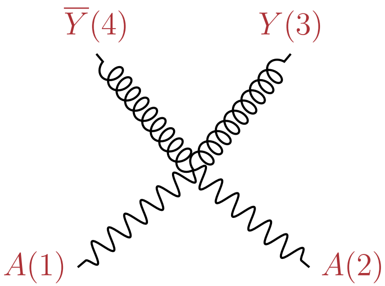
$$= \frac{1}{\sqrt{2}}[ig(p_1 - p_2)^{\mu_3}\eta_{\mu_1\mu_2} + ig(p_2 - p_3)^{\mu_1}\eta_{\mu_2\mu_3} + ig(p_3 - p_1)^{\mu_2}\eta_{\mu_1\mu_3}]$$

4.3 4-point vertex functions

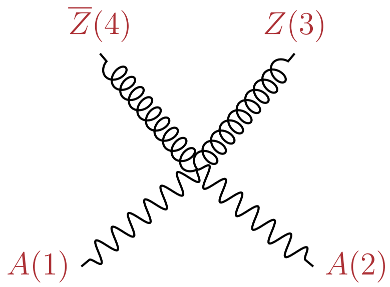
Here are four point vertices (17)

- 

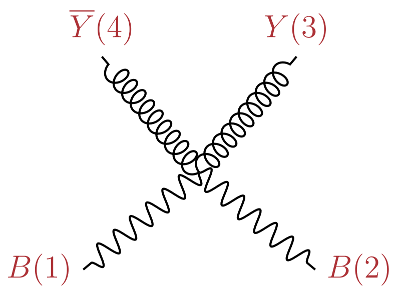
$$-2ig^2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$$

- 

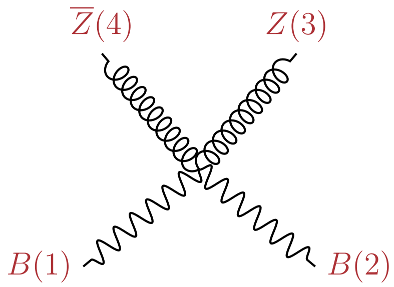
$$-\frac{1}{2}ig^2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$$

- 

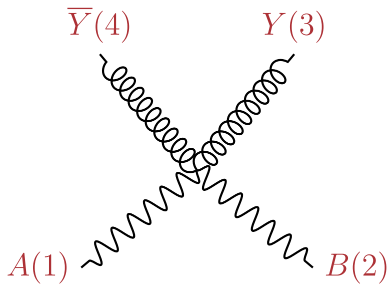
$$-\frac{1}{2} i g^2 \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4}$$

- 

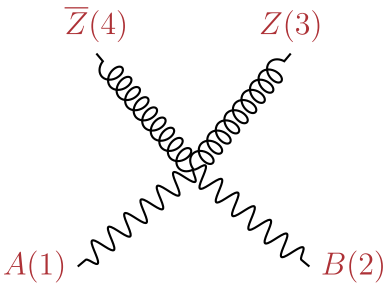
$$-\frac{3}{2} i g^2 \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4}$$

- 

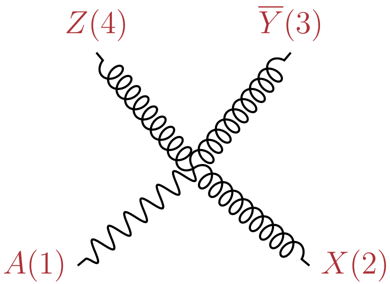
$$-\frac{3}{2} i g^2 \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$$

- 

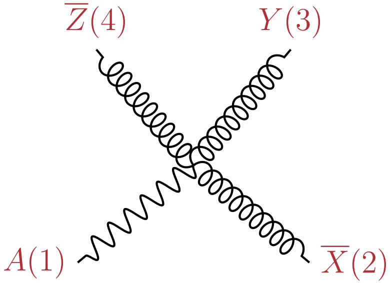
$$-\frac{\sqrt{3}}{2} i g^2 \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$$

- 

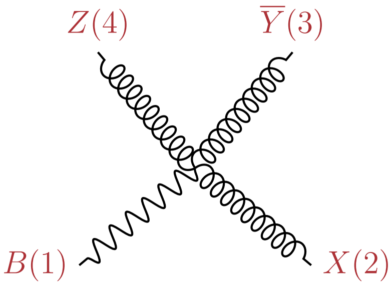
$$\frac{\sqrt{3}}{2} i g^2 \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}$$

- 

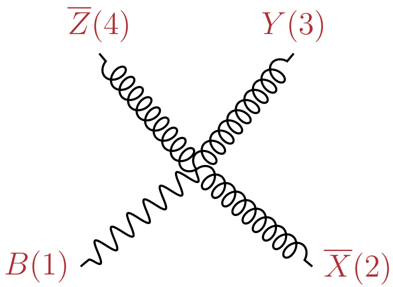
$$\frac{3}{2\sqrt{2}} i g^2 [\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}]$$

- 

$$\frac{3}{2\sqrt{2}} ig^2 [\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}]$$

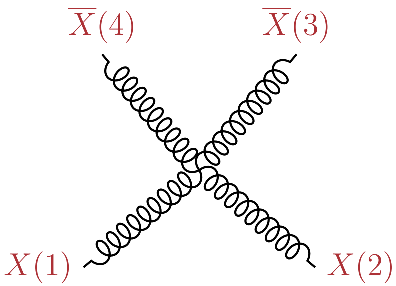
- 

$$\frac{\sqrt{3}}{2\sqrt{2}} ig^2 [\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - 2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}]$$

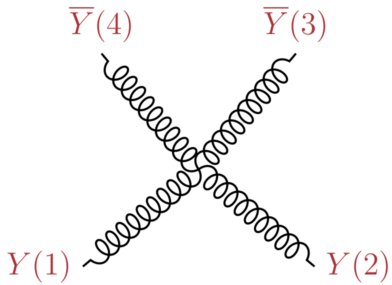
• 

$$\frac{\sqrt{3}}{2\sqrt{2}} i g^2 [\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - 2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}]$$

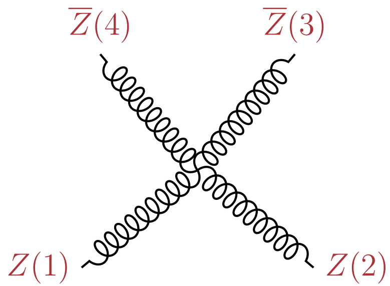
4.4 quartic vertices of second kind:

• 

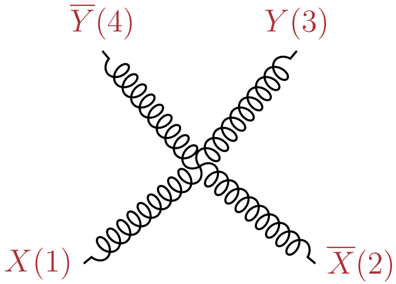
$$-ig^2 [\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - 2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}]$$

- 

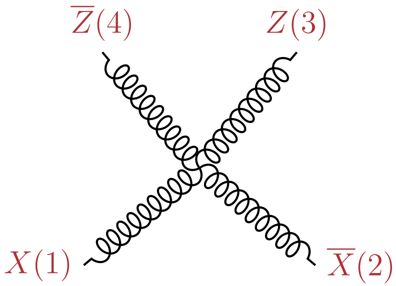
$$-ig^2[\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - 2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}]$$

- 

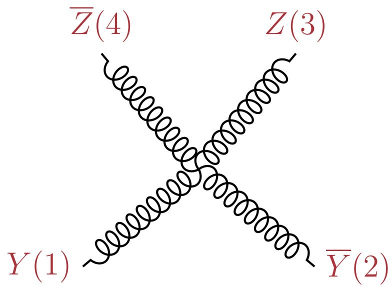
$$-ig^2[\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - 2\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}]$$

- 

$$-\frac{1}{2}ig^2\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}+ig^2\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}$$

- 

$$+ig^2\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}$$

- 

$$+ig^2\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}$$

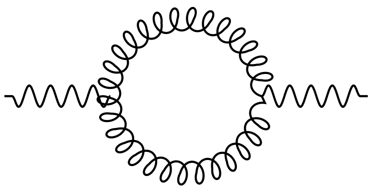
5 One loop divergences: beta function

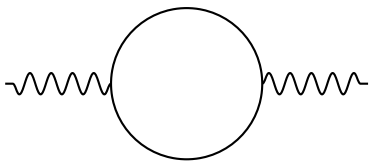
5.1 direct evaluations of Feynman diagrams

- Abelian projected QCD shares a common feature with QED: Ward-Takahash identity. The wave function renormalization of charged particle is canceled by the vertex function renormalization.

$$Z_2 = Z_1$$

And thus charge renormalization is determined by photon wave function renormalization.





5.2 Heat Kernel Expansion

The one loop effective action is formally

$$\exp(-i\Gamma[A]) = \frac{\sqrt{\mathrm{Det} M_V}}{\sqrt{\mathrm{Det} M_F}\ \mathrm{Det} M_{ghost}}$$

where

$$\begin{aligned} M_V &= \eta^\mu_\nu D^2 - 2igF^\mu_\nu \\ M_F &= \not{D}\not{D} = D^2 - 2ig\sigma_{\mu\nu}F^{\mu\nu} \\ M_{ghost} &= D^2 \end{aligned}$$

The functional determinant is expressed as, using propertime representation

$$\ln \mathrm{Det} M = \mathrm{Tr} \ln M = \mathrm{Tr} \int \frac{ds}{s} e^{-isD}$$

The Heat kernel allows an asymptotic expansion

$$\Delta x, y, s = \langle x|e^{-isD}|y\rangle = \frac{e^{-(x-y)^2/is}}{(4\pi is)^2}\sum_k a_k(x,y)(is)^k$$

Obviously, UV divergencies are controlled by the second coefficients $a_2(x,x)$, which can be easily found by recurrence relations.

5.3 Nielsen-Hughes formula

One may evaluate the functional determinant by finding all eigenvalues, when the background field corresponds to constant uniform magnetic field. This expression also has an UV divergent part. We may identify beta function from it. For a charged field of any spin S,

$$\beta_0 = -\frac{(-1)^{2S}}{2\pi}[(2S)^2 - \frac{1}{3}]$$

We may count the number of charged vectors and quarks and have found

$$\beta_0 = -\frac{1}{6\pi}\left(\frac{33}{2} - N_f\right)$$

5.4 All order proof

It was prove that this “modified” YM theory is renormalizable in all orders in perturbation theory, based on BRS structure of the system.

5.5 Other possible application

Since Feynman rules are modified, we may find physical processes showing some Abelian nature.